



TITLE:

ティープリッツ作用素が正規または解析的になる為の条件 (作用素および作用素不等式の最近の話題)

AUTHOR(S):

Yoshino, Takashi

CITATION:

Yoshino, Takashi. ティープリッツ作用素が正規または解析的になる為の条件 (作用素および作用素不等式の最近の話題). 数理解析研究所講究録 2002, 1259: 42-46

ISSUE DATE:

2002-04

URL:

<http://hdl.handle.net/2433/41963>

RIGHT:

ティープリッツ作用素が正規または解析的になる為の条件

東北大学・大学院理学研究科 吉野 崇 (Takashi Yoshino)
Mathematical Institute, Tôhoku University

For a bounded measurable function $\varphi \in L^\infty$ on the unit circle, Toeplitz operator T_φ is defined by $T_\varphi f = P\varphi f$ for $f \in H^2$, where P is the orthogonal projection from L^2 onto H^2 . If $\varphi \in H^\infty$, then we say that T_φ is analytic. Halmos asks whether every subnormal Toeplitz operator is either analytic or normal. And concerning this problem Amemiya-Itô-Wong prove that every quasi-normal Toeplitz operator is only normal or a scalar multiple of an isometry. We shall give here a condition that the Toeplitz operator T_φ is normal or analytic.

The following results are well known.

Proposition 1. ([2]) If \mathcal{M} is a non-zero invariant subspace of T_z , then there exists an isometric Toeplitz operator T_g such that $\mathcal{M} = T_g H^2$.

Proposition 2. ([3]) $A \in \mathcal{B}(H^2)$ is a Toeplitz operator if and only if $T_z^* A T_z = A$. And, in particular, $A \in \mathcal{B}(H^2)$ is an analytic Toeplitz operator if and only if $T_z A = A T_z$.

Proposition 3. ([3]) $T_\varphi T_\psi$ is a Toeplitz operator if and only if $\bar{\varphi}$ or $\psi \in H^\infty$. In this case, $T_\varphi T_\psi = T_{\varphi\psi}$.

Proposition 4. ([4]) If φ is a non-constant function in L^∞ , then $\sigma_p(T_\varphi) \cap \overline{\sigma_p(T_\varphi^*)} = \emptyset$ where $\sigma_p(T_\varphi)$ denotes the point spectrum of T_φ .

Lemma 1. For any $\varphi \notin H^\infty$, T_φ has no such type of invariant subspace as $T_g H^2$ for some non-constant inner function g .

Proof. For some non-constant inner function g , if $T_\varphi T_g H^2 \subseteq T_g H^2$, then there exists a $C \in \mathcal{B}(H^2)$ such that $T_{\varphi g} = T_g C$ because $T_\varphi T_g = T_{\varphi g}$ by Proposition 3. Since g is inner, $C = T_g^* T_{\varphi g} = T_\varphi$ and $T_{\varphi g} = T_g T_\varphi$ and hence $\varphi \in H^\infty$ by Proposition 3 because $\bar{g} \notin H^\infty$.

Lemma 2. For $\varphi \in H^\infty$, if $(T_\varphi^* T_\varphi)^2 = T_\varphi^{*2} T_\varphi^2$, then φ is a scalar multiple of an inner function.

Proof. By Proposition 3 and by the assumption,

$$T_{\bar{\varphi}\varphi}^2 = (T_\varphi^* T_\varphi)^2 = T_\varphi^{*2} T_\varphi^2 = T_{\bar{\varphi}^2} T_{\varphi^2} = T_{\bar{\varphi}^2 \varphi^2} = T_{|\varphi|^4}$$

and $\bar{\varphi}\varphi \in H^\infty$ and hence $|\varphi|$ is constant. Therefore φ is a scalar multiple of an inner function.

For $\varphi \in L^\infty$, let $X_\varphi = T_\varphi T_z - T_z T_\varphi$ and let $Y_\varphi = T_z^* T_\varphi^* T_\varphi T_z - T_\varphi^* T_\varphi$. Then

$$X_\varphi = O \Rightarrow \varphi \in H^\infty \text{ by Proposition 2,}$$

$$Y_\varphi = O \Rightarrow T_\varphi^* T_\varphi \text{ is a Toeplitz operator by Proposition 2}$$

$$\Rightarrow \varphi \in H^\infty \text{ by Proposition 3,}$$

$$\text{and } Y_\varphi = T_z^* T_\varphi^* (T_z T_\varphi + X_\varphi) - T_\varphi^* T_\varphi = T_z^* T_\varphi^* X_\varphi.$$

Since $Y_\varphi = T_z^* T_\varphi^* (I - T_z T_z^*) T_\varphi T_z$ and since $(I - T_z T_z^*) H^2 = \vee\{1\}$, Y_φ is an at most rank one positive operator and $Y_\varphi T_z^* T_\varphi^* 1 = \|Y_\varphi\| T_z^* T_\varphi^* 1$.

And since, for any $f \in H^2$, $\|X_\varphi f\|_2^2 = \|(I - T_z T_z^*) T_\varphi T_z f\|_2^2 = \langle Y_\varphi f, f \rangle = \|Y_\varphi^{\frac{1}{2}} f\|_2^2$, $\mathcal{N}_{X_\varphi} = \mathcal{N}_{Y_\varphi}$ and $X_\varphi^* H^2 = Y_\varphi H^2 = \vee\{T_z^* T_\varphi^* 1\}$ and hence

$$\begin{aligned} H^2 &= \{f \in H^2 : Y_\varphi f = 0\} \oplus \{f \in H^2 : Y_\varphi f = \|Y_\varphi\| f\} \\ &= \mathcal{N}_{X_\varphi} \oplus \vee\{T_z^* T_\varphi^* 1\} \end{aligned} \quad (\#)$$

and also we have $X_\varphi H^2 \subseteq \mathcal{N}_{T_z^*} = \vee\{1\}$.

Lemma 3. If $\{0\} \neq \mathcal{N}_{T_\varphi^* T_\varphi - T_\varphi T_\varphi^*} \neq H^2$, then $Y_\varphi - Y_{\bar{\varphi}} \neq O$ and $(Y_\varphi - Y_{\bar{\varphi}}) H^2 = \vee\{T_z^* T_\varphi^* 1, T_z^* T_{\bar{\varphi}} 1\}$.

Proof. If $Y_\varphi - Y_{\bar{\varphi}} = O$, then $T_\varphi^* T_\varphi - T_\varphi T_\varphi^*$ is a Hermitian Toeplitz operator by Proposition 2 because $Y_\varphi - Y_{\bar{\varphi}} = T_z^* (T_\varphi^* T_\varphi - T_\varphi T_\varphi^*) T_z - (T_\varphi^* T_\varphi - T_\varphi T_\varphi^*)$. Let $T_\varphi^* T_\varphi - T_\varphi T_\varphi^* = T_\psi$. Then the assumption implies $\psi \neq 0$ and $0 \in \sigma_p(T_\psi)$. This contradicts Proposition 4. And since, for any $f \in H^2$,

$$(Y_\varphi - Y_{\bar{\varphi}})f = \langle f, T_z^* T_\varphi^* 1 \rangle \|Y_\varphi\| T_z^* T_\varphi^* 1 - \langle f, T_z^* T_{\bar{\varphi}} 1 \rangle \|Y_{\bar{\varphi}}\| T_z^* T_{\bar{\varphi}} 1,$$

we have $(Y_\varphi - Y_{\bar{\varphi}}) H^2 = \vee\{T_z^* T_\varphi^* 1, T_z^* T_{\bar{\varphi}} 1\}$.

Theorem. If T_φ satisfies the following conditions ; (i) $(T_\varphi^* T_\varphi)^2 = T_\varphi^{*2} T_\varphi^2$, (ii) $\{0\} \neq \mathcal{N}_{T_\varphi^* T_\varphi - T_\varphi T_\varphi^*}$, (iii) Every eigen-space of $T_\varphi^* T_\varphi$ is invariant under T_φ^* and

(iv) $T_\varphi^*T_z^*T_\varphi^*1$ and $T_\varphi^*T_z^*T_\varphi 1$ are linearly dependent, then T_φ is normal or a scalar multiple of an isometry.

Proof. By Lemma 2, we have only to prove that there is no non-normal, non-analytic Toeplitz operator which satisfies the conditions (i), (ii), (iii) and (iv).

Let T_φ be non-normal and non-analytic. Since $T_\varphi^*(T_\varphi^*T_\varphi - T_\varphi T_\varphi^*)T_\varphi = O$ by (i),

$$\begin{aligned} T_\varphi^*(Y_\varphi - Y_{\bar{\varphi}})T_\varphi &= T_\varphi^*T_z^*(T_\varphi^*T_\varphi - T_\varphi T_\varphi^*)T_zT_\varphi \\ &= (T_z^*T_\varphi^* - X_\varphi^*)(T_\varphi^*T_\varphi - T_\varphi T_\varphi^*)(T_\varphi T_z - X_\varphi) \\ &= -T_z^*T_\varphi^*(T_\varphi^*T_\varphi - T_\varphi T_\varphi^*)X_\varphi - X_\varphi^*(T_\varphi^*T_\varphi - T_\varphi T_\varphi^*)(T_\varphi T_z - X_\varphi) \end{aligned} \quad (1)$$

and $T_z^*T_\varphi^*(T_\varphi^*T_\varphi - T_\varphi T_\varphi^*)X_\varphi H^2 \subseteq X_\varphi^*H^2 + T_\varphi^*(Y_\varphi - Y_{\bar{\varphi}})H^2$ and hence, by Lemma 3,

$$\begin{aligned} T_z^*T_\varphi^*(T_\varphi^*T_\varphi - T_\varphi T_\varphi^*)1 &= \alpha T_z^*T_\varphi^*1 + \beta T_\varphi^*T_z^*T_\varphi^*1 + \gamma T_\varphi^*T_z^*T_\varphi 1 \\ &\text{for some } \alpha, \beta, \gamma \in \mathbb{C} \end{aligned} \quad (2)$$

because the conditions of Lemma 3 are satisfied by (ii) and by the non-normality of T_φ . And since

$$\begin{aligned} T_z^*(T_\varphi^*T_\varphi - T_\varphi T_\varphi^*)1 &= (T_\varphi^*T_z^* + X_\varphi^*)T_\varphi 1 - (T_\varphi T_z^* + X_{\bar{\varphi}}^*)T_\varphi^*1 \\ &= T_\varphi^*T_z^*T_\varphi 1 + aT_z^*T_\varphi^*1 - T_\varphi T_z^*T_\varphi^*1 + bT_z^*T_\varphi 1 \\ &\text{for some } a, b \in \mathbb{C}, \end{aligned}$$

$$\begin{aligned} &T_z^*T_\varphi^*(T_\varphi^*T_\varphi - T_\varphi T_\varphi^*)1 \\ &= (T_\varphi^*T_z^* + X_\varphi^*)(T_\varphi^*T_\varphi - T_\varphi T_\varphi^*)1 \\ &= T_\varphi^*T_z^*(T_\varphi^*T_\varphi - T_\varphi T_\varphi^*)1 + X_\varphi^*(T_\varphi^*T_\varphi - T_\varphi T_\varphi^*)1 \\ &= T_\varphi^*T_z^*(T_\varphi^*T_\varphi - T_\varphi T_\varphi^*)1 + cT_z^*T_\varphi^*1 \quad \text{for some } c \in \mathbb{C} \\ &= T_\varphi^*(T_\varphi^*T_z^*T_\varphi 1 + aT_z^*T_\varphi^*1 - T_\varphi T_z^*T_\varphi^*1 + bT_z^*T_\varphi 1) + cT_z^*T_\varphi^*1 \\ &= T_\varphi^{*2}T_z^*T_\varphi 1 + aT_\varphi^*T_z^*T_\varphi^*1 - T_\varphi^*T_\varphi T_z^*T_\varphi^*1 + bT_\varphi^*T_z^*T_\varphi 1 + cT_z^*T_\varphi^*1 \end{aligned}$$

and, by (2),

$$\begin{aligned} &T_\varphi^*T_\varphi T_z^*T_\varphi^*1 \\ &= T_\varphi^{*2}T_z^*T_\varphi 1 + (c - \alpha)T_z^*T_\varphi^*1 + (a - \beta)T_\varphi^*T_z^*T_\varphi^*1 + (b - \gamma)T_\varphi^*T_z^*T_\varphi 1. \end{aligned} \quad (3)$$

Since, by (1),

$$T_\varphi^*(Y_\varphi - Y_{\bar{\varphi}})T_\varphi H^2 \subseteq T_z^*T_\varphi^*(T_\varphi^*T_\varphi - T_\varphi T_\varphi^*)X_\varphi H^2 + X_\varphi^* H^2 \quad (4)$$

and since $T_\varphi^*(Y_\varphi - Y_{\bar{\varphi}})T_\varphi H^2 \subseteq T_\varphi^*(Y_\varphi - Y_{\bar{\varphi}})H^2$,

$$\begin{aligned} T_\varphi^*T_z^*T_\varphi^*1 &= \lambda_1 T_z^*T_\varphi^*1 \\ \text{and } T_\varphi^*T_z^*T_\varphi 1 &= \lambda_2 T_z^*T_\varphi^*1 \\ \text{for some } \lambda_1, \lambda_2 &\in \mathbb{C} \end{aligned} \quad (5)$$

by (iv), Lemma 3 and (2). And hence, by (3),

$$T_\varphi^*T_\varphi(T_z^*T_\varphi^*1) = \{\lambda_2\lambda_1 + (c-\alpha) + (a-\beta)\lambda_1 + (b-\gamma)\lambda_2\}T_z^*T_\varphi^*1. \quad (6)$$

Let $r = \lambda_2\lambda_1 + (c-\alpha) + (a-\beta)\lambda_1 + (b-\gamma)\lambda_2$ and $\mathcal{M} = \{f \in H^2 : T_\varphi^*T_\varphi f = rf\}$.

Since, for any $f \in \mathcal{M}$,

$$\begin{aligned} (T_\varphi^*T_\varphi - rI)T_z^*f &= T_\varphi^*(T_z^*T_\varphi - X_{\bar{\varphi}}^*)f - rT_z^*f \\ &= (T_z^*T_\varphi^* - X_\varphi^*)T_\varphi f - T_\varphi^*X_{\bar{\varphi}}^*f - rT_z^*f \\ &= -X_\varphi^*T_\varphi f - T_\varphi^*X_{\bar{\varphi}}^*f \\ &= -a_1T_z^*T_\varphi^*1 - T_\varphi^*(b_1T_z^*T_\varphi 1) \quad \text{for some } a_1, b_1 \in \mathbb{C} \\ &= -(a_1 + b_1\lambda_2)T_z^*T_\varphi^*1 \quad \text{by (5)} \end{aligned}$$

and since $T_z^*T_\varphi^*1 \in \mathcal{M}$ by (6), $(T_\varphi^*T_\varphi - rI)^2T_z^*f = 0$ and $(T_\varphi^*T_\varphi - rI)T_z^*f = 0$ because $\|(T_\varphi^*T_\varphi - rI)T_z^*f\|_2^2 = \langle (T_\varphi^*T_\varphi - rI)^2T_z^*f, T_z^*f \rangle = 0$ and hence \mathcal{M} is invariant under T_z^* . Since T_φ is non-analytic by the assumption, $T_z^*T_\varphi^*1 \neq 0$ by (‡) and by Proposition 2 and $\mathcal{M} \neq H^2$ by Proposition 3 and hence \mathcal{M} is non-trivial. Therefore $\mathcal{M}^\perp = T_g H^2$ for some non-constant inner function g by Proposition 1. Since \mathcal{M} is invariant under T_φ^* by (iii), $T_g H^2$ is invariant under T_φ and $\varphi \in H^\infty$ by Lemma 1. This contradicts the assumption that T_φ is non-analytic.

Corollary. ([1]) Every quasi-normal T_φ (i.e., T_φ commutes with $T_\varphi^*T_\varphi$) is only normal or a scalar multiple of an isometry.

Proof. It is clear that every quasi-normal T_φ satisfies the conditions (i), (ii) and (iii). And, by Theorem, we have only to show that quasi-normal T_φ satisfies the condition (iv).

If $T_\varphi^* T_z^* T_\varphi^* 1$ and $T_\varphi^* T_z^* T_\varphi 1$ are linearly independent, then

$$(Y_\varphi - Y_{\bar{\varphi}})T_\varphi H^2 = \vee\{T_z^* T_\varphi^* 1, T_z^* T_\varphi 1\}$$

because, for any $f \in H^2$,

$$\begin{aligned} (Y_\varphi - Y_{\bar{\varphi}})T_\varphi f &= \langle T_\varphi f, T_z^* T_\varphi^* 1 \rangle \|Y_\varphi\| T_z^* T_\varphi^* 1 - \langle T_\varphi f, T_z^* T_\varphi 1 \rangle \|Y_{\bar{\varphi}}\| T_z^* T_\varphi 1 \\ &= \langle f, T_\varphi^* T_z^* T_\varphi^* 1 \rangle \|Y_\varphi\| T_z^* T_\varphi^* 1 - \langle f, T_\varphi^* T_z^* T_\varphi 1 \rangle \|Y_{\bar{\varphi}}\| T_z^* T_\varphi 1. \end{aligned}$$

And since $T_\varphi^*(Y_\varphi - Y_{\bar{\varphi}})T_\varphi H^2 \subseteq X_\varphi^* H^2$ by (4) in the proof of Theorem because $T_\varphi^*(T_\varphi^* T_\varphi - T_\varphi T_\varphi^*) = O$ by the quasi-normality of T_φ ,

$$T_\varphi^* T_z^* T_\varphi^* 1 = \lambda_1 T_z^* T_\varphi^* 1$$

$$\text{and } T_\varphi^* T_z^* T_\varphi 1 = \lambda_2 T_z^* T_\varphi^* 1 \quad \text{for some } \lambda_1, \lambda_2 \in \mathbb{C}$$

and this contradicts the assumption that $T_\varphi^* T_z^* T_\varphi^* 1$ and $T_\varphi^* T_z^* T_\varphi 1$ are linearly independent.

References

- [1] I. Amemiya, T. Ito and T. K. Wong,
On quasinormal Toeplitz operators,
Proc. Amer. Math. Soc., 50(1975), 254–258.
- [2] A. Beurling,
On two problems concerning linear transformations in Hilbert space,
Acta Math., 81(1949), 239–255.
- [3] A. Brown and P. R. Halmos,
Algebraic properties of Toeplitz operators,
J. Reine Angew. Math., 213(1964), 89–102.
- [4] T. Yoshino,
Note on Toeplitz operators,
Tohoku Math. Journ., 26(1974), 535–540.